

A DUALITY BETWEEN CERTAIN SPHERES AND ARCS IN S^3

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1. **Introduction.** Harrold and Moise [18] have shown that if a 2-sphere in the standard 3-sphere S^3 is locally polyhedral except at one point, then one of its closed complementary domains is a closed 3-cell. Cantrell [6] has shown that the other open complementary domain is an open 3-cell and [8], [9] that if an $(n-1)$ -sphere Σ in the standard n -sphere S^n is locally flat (see [4] and [5, p. 49] for definitions) except at one point for $n > 3$, then Σ is flat in S^n . Fox and Artin [13] have given the first examples of arcs which are locally flat except at one point.

The main result of this paper is a duality between 2-spheres which are locally flat in S^3 except at one point and arcs which are locally flat in S^3 except at one endpoint. Roughly, if Σ is a 2-sphere which is locally flat in S^3 except possibly at one point p , then we associate with Σ any arc in Σ which has p for an endpoint. Conversely, if α is an arc which is locally flat in S^3 except possibly at one endpoint p , then we "blow up" α into a little 2-sphere which tapers down to p just as α does and we associate this sphere with α . We make this precise in §3.

We extend this result to a duality theorem concerning nearly flat 2-manifolds in a 3-manifold. As an application of this duality theorem, in §4 we prove a uniqueness theorem in a class of decomposition spaces. In §5 we extend a result of Lininger [22] by characterizing a class of crumpled cubes.

In §6 pseudo-half spaces are characterized. An n -pseudo-half space M^n is an n -manifold with boundary such that the interior of M^n is homeomorphic to R^n and the boundary of M^n is homeomorphic to R^{n-1} . Cantrell [7] and Doyle [11] have shown that, for $n \neq 3$, every n -pseudo-half space is homeomorphic to the closed half-space R_+^n . Kwun and Raymond [21] give an example of a 3-pseudo-half space which is not homeomorphic to the closed half-space R_+^3 . It follows from [1], [14] that uncountably many topologically different 3-pseudo-half spaces exist. In Theorem 7 we prove the following:

M^n is an n -pseudo-half space if and only if M^n is homeomorphic to $B^n - \alpha$ where α is arc in the standard closed n -ball B^n such that α intersects its boundary S^{n-1} at one endpoint and $S^{n-1} \cup \alpha$ is locally flat except possibly at the other endpoint.

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Harrold [17] has given a sufficient condition for an arc in S^3 to be cellular (see [3] for definition) and Doyle [12] has given a sufficient condition for an arc in S^n to be cellular. McMillan [23] has shown that, for $n \neq 4$, a subarc of a cellular arc is cellular. Stewart [26] has given an example of a cellular arc in S^3 which is wild at every point. In §7 we prove the following:

If α is an arc in S^3 such that α contains a subarc β both of whose endpoints are isolated wild points of β , then α is not cellular.

2. Preliminary results. If X is locally flat at point x in a triangulated n -manifold N^n , then X is locally tame at x . Thus it follows from Bing's Approximation Theorem [2] that if X , a closed subset of a triangulated 3-manifold N^3 , is locally flat except on a set Y , then X is equivalent to a set K which contains Y such that $K - Y$ is locally polyhedral. Hence if a 2-sphere Σ in S^3 is locally flat except at one point, then by [18] one of the closed complementary domains of Σ is a closed 3-cell. Moreover, if X is a 2-sphere or an arc in a 3-manifold N^3 , then the following statements are equivalent:

- (1) X is locally flat at x ,
- (2) X is locally tame at x .

Also we will use the facts established in [3], [4] that if Σ is an $(n-1)$ -sphere in S^n , then the following statements are equivalent:

- (1) Σ is locally flat at every point of Σ ,
- (2) Σ is flat,
- (3) Σ is bi-collared.

The two theorems in this section seem to be folk theorems in this subject. The proof of Theorem 1 is standard but the proof of Theorem 2 is often incomplete so that we will include it here.

THEOREM 1. *Let α and β be arcs in an n -manifold M which are locally flat except at the common endpoint p such that α is a proper subarc of β and let U be a neighborhood of $\beta - p$. Then there is a pseudo-isotopy $\phi_t (t \in I)$ of M onto itself such that:*

- (1) $\phi_0 = 1$,
- (2) $\phi_t|_{(M-U)} = 1$,
- (3) $\phi_1(\alpha) = p$,
- (4) $\phi_1(\beta) = \alpha$,
- (5) $\phi_1|_{(M-\alpha)}$ is a homeomorphism onto $M - p$.

THEOREM 2. *Let M be a manifold with boundary F . Add a closed collar $F \times [-1, 0]$ to M by identifying $(x, 0)$ with x for $x \in F$. Then $M \cup (F \times [-1, 0]) \approx M$ and $M \cup (F \times (-1, 0]) \approx \text{Int } M$.*

Proof. By Theorem 2 of [4] F is collared in M , so that there is a homeomorphism $H: F \times [-1, 1] \rightarrow M \cup (F \times [-1, 0])$ such that:

$$H(x, t) = (x, t), \quad x \in F, t \in [-1, 0],$$

$$H(x, t) \in M, \quad x \in F, t \in [0, 1].$$

Let $Y = H(F \times 1)$. If F is compact, then Y is closed in M and

$$H(F \times [-1, 1]) \cap (M - H(F \times I')) = Y,$$

where $I' = [0, 1]$. Thus there is a canonical map which pushes F out to $F \times \{-1\}$ and which is the identity on $M - H(F \times I')$. However, if F is not compact, Y may not be closed in M nor separate M .

For each $x \in F$, let $\delta_x = \frac{1}{2}\rho(x, M - H(F \times I'))$, where ρ is a metric in M . Then $U = \bigcup_{x \in F} V_{\delta_x}(x)$ is a neighborhood of F in M and the triangle inequality insures that $\text{Cl } U \subset H(F \times I')$.

Given a map $\lambda: F \rightarrow (0, 1]$, we define the spindle neighborhood $S(F, \lambda)$ by:

$$S(F, \lambda) = \{(x, t) \in F \times I' \mid x \in F \text{ and } t < \lambda(x)\}.$$

By [4] the spindle neighborhoods form a neighborhood basis for $F \times 0$ in $F \times I'$. So let $S(F, \lambda)$ be a spindle neighborhood of $F \times 0$ such that $S(F, \lambda) \subset H^{-1}(U)$. Define $G: F \times [-1, 1] \rightarrow M \cup (F \times [-1, 0])$ by $G(x, t) = H(x, t\lambda(x))$ and let $X = G(F \times 1)$. Then X is closed in M and

$$G(F \times [-1, 1]) \cap (M - G(F \times I')) = X.$$

Thus there is a canonical map which pushes F out to $F \times \{-1\}$ and which is the identity on $M - G(F \times I')$. Moreover, $\text{Int } M$ is mapped onto $M \cup (F \times (-1, 0])$.

We conclude this section with several lemmas.

LEMMA 1. *Let K be a disk in R^3 that is locally polyhedral except at an interior point p . Then there is a polyhedral disk D with boundary F such that $D \cap K = F$ and F separates p from $\text{Bd } K$ in K .*

Proof. This is a generalization of Lemma 1 of [18] and the proof is essentially the same.

LEMMA 2. *Let K be a disk in R^3 that is locally flat except at an interior point p . Then there is a homeomorphism $g: B^3 \rightarrow R^3$ such that:*

- (1) $g(S_+^2) \subset K$,
- (2) $g(0, 0, 1) = p$,
- (3) $g(B^3 - S_+^2) \subset R^3 - K$,
- (4) $g(S^2)$ is locally flat except at p ,
- (5) $(K - g(S_+^2)) \cup g(S_-^2)$ is locally flat.

Proof. It follows from a remark above that we can assume with no loss of generality that K is locally polyhedral except at p . By Lemma 1 there is a polyhedral disk D with boundary F such that $D \cap K = F$ and F separates p from $\text{Bd } K$ in K . Let P be the closed complementary domain of F in K such that P contains p

Since the 2-sphere $\Sigma = P \cup D$ is locally polyhedral except at p , it follows from Theorem 1 of [18] that Σ is collared in one of the closed complementary domains of Σ in R^3 . This establishes the existence of g with the required properties.

Let B^n be the closed unit n -ball centered at the origin in R^n and let $B_r(x) = \text{Cl}V_r(x)$ be the closed n -ball of radius r centered at x . For the rest of this section and in Theorem 4, we will use the following definitions:

$$\begin{aligned} a &= (0, 0, 1), \\ b &= (0, 0, -1), \\ J &= [(0, 0, 1/2), (0, 0, 1)], \\ D_0 &= \{(x, y, z) \in B^3 \mid z = 1/2\}, \\ G_0 &= \{(x, y, z) \in B^3 \mid z \geq 1/2\}, \\ P_0 &= \{(x, y, z) \in S^2 \mid z \geq 1/2\}. \end{aligned}$$

LEMMA 3. Let $\varepsilon > 0$, let

$$A = (\text{Bd}B_2(b)) \cup \{(x, y, z) \in B_2(b) \mid z < -\varepsilon\},$$

and let $d \in \text{Int}D_0$. Then there is a homeomorphism h of $B_2(b)$ onto itself such that:

- (1) $h|_A = 1$,
- (2) $h(S^2) = S^2$,
- (3) $h(D_0) = B^2$,
- (4) $h(d) = 0$.

LEMMA 4. Let

$$A = (\text{Bd}B_2(b)) \cup (R_-^3 \cap (B_2(b) - V_1(0))).$$

Then there is a map ϕ of $B_2(b)$ onto itself such that:

- (1) $\phi|_A = 1$,
- (2) $\phi(0 \times B_+^1) = a$,
- (3) $\phi(B^2) = S_+^2$,
- (4) $\phi|(B_2(b) - (0 \times B_+^1))$ is a homeomorphism onto $B_2(b) - a$.

LEMMA 5. Let G be a closed 3-cell and let $f_1, f_2: B \twoheadrightarrow G$ (" $\twoheadrightarrow$ " means "onto") be homeomorphisms such that $f_1(a) = f_2(a)$ and $f_1(b) = f_2(b)$. Then there is a homeomorphism $h: G \twoheadrightarrow G$ such that:

- (1) $h|_{\text{Bd}G} = 1$,
- (2) $hf_1|(0 \times B^1) = f_2|(0 \times B^1)$.

Proof. Define $g: S^2 \twoheadrightarrow S^2$ by $g = f_2^{-1}f_1|_{S^2}$. Extend it to a homeomorphism $g: B^3 \twoheadrightarrow B^3$ by radial extension. Then $g|(0 \times B^1) = 1$. Define $h: G \twoheadrightarrow G$ by $h = f_2gf_1^{-1}$. Then

$$\begin{aligned} h|_{\text{Bd}G} &= f_2gf_1^{-1}|_{\text{Bd}G} = f_2(f_2^{-1}f_1)f_1^{-1}|_{\text{Bd}G} = 1, \\ hf_1|(0 \times B^1) &= (f_2gf_1^{-1})f_1|(0 \times B^1) = f_2g|(0 \times B^1) \\ &= f_2|(0 \times B^1). \end{aligned}$$

LEMMA 6. If Σ_1 and Σ_2 are disjoint 2-spheres in S^3 , $A = [\Sigma_1, \Sigma_2]$ and α is an arc in S^3 such that $\Sigma_i \cap \alpha = p_i$, a point, and $\Sigma_i \cup \alpha$ is locally flat, $i = 1, 2$, then $(A, A \cap \alpha) \approx (S^2 \times I, 1 \times I)$.

Proof. We identify S^3 with the one-point compactification of R^3 . Let

$$C = B^2 \times [0, 1] \cup (0 \times [1, 2]) \cup B^2 \times [2, 3].$$

Let G_i be the closed complementary domain of Σ_i which does not contain A , $i = 1, 2$. Let $f: C \rightarrow G_1 \cup \alpha \cup G_2$ be a homeomorphism such that $f(B^2 \times [0, 1]) = G_1$, $f(0 \times [1, 2]) = \alpha \cap A$, $f(B^2 \times [2, 3]) = G_2$ and so that $f|_{(B^2 \times [0, 1])}$ and $f|_{(B^2 \times [2, 3])}$ induce the same orientation on S^3 . Evidently f is a locally flat embedding of C into S^3 .

It follows from the Annulus Theorem in S^3 (see for example [24], [15]) that there is a stable homeomorphism $h: S^3 \rightarrow S^3$ such that $hf(x, t) = f(x, t + 2)$ for all $x \in B^2$, $t \in I$. It follows from Lemma 7.1 of [5] that there is a homeomorphism $g: S^3 \rightarrow S^3$ such that gf is the inclusion $C \subset R^3 \subset S^3$. Let g_1 be a homeomorphism of $\text{Cl}(S^3 - C)$ onto $S^2 \times I$ such that $g_1(0 \times [1, 2]) = 1 \times I$. Then $g_1(g|_A)$ is a homeomorphism of $(A, A \cap \alpha)$ onto $(S^2 \times I, 1 \times I)$.

LEMMA 7. If Σ_1 and Σ_2 are flat 2-spheres in S^3 and α is an arc in S^3 such that $\Sigma_i \cap \alpha = q$, an endpoint of α , and $\Sigma_i \cup \alpha$ is locally flat at q , $i = 1, 2$, then there is a homeomorphism $h: S^3 \rightarrow S^3$ such that $h(\Sigma_1) = \Sigma_2$ and $h(\alpha) = \alpha$.

Proof. Since Σ_1 and Σ_2 are flat and α is locally flat at q , there is a flat 2-sphere Σ_3 in S^3 such that $\Sigma_1 \cup \Sigma_2 \subset E$, an open complementary domain of Σ_3 , $\Sigma_3 \cap \alpha = p$, an interior point of α , $\Sigma_3 \cup \alpha$ is locally flat at p and $\beta = \alpha \cap [\Sigma_3, \Sigma_1]$ is locally flat. By Lemma 6 there is a homeomorphism $h: ([\Sigma_3, \Sigma_1], \beta) \rightarrow ([\Sigma_3, \Sigma_2], \beta)$. Without loss of generality $h|_{\Sigma_3} = 1$, so we can extend h to $S^3 - E$ by the identity. Since Σ_1 and Σ_2 are flat, their closed complementary domains are closed 3-cells and we can extend h to a homeomorphism of S^3 onto itself with the desired properties.

3. Duality theorems. Let $\mathcal{J}$ be the set of pairs (α, p) where α is an arc in S^3 and p is an endpoint of α such that α is locally flat except possibly at p and let $\mathcal{S}$ be the set of pairs (Σ, p) where Σ is a 2-sphere in S^3 and p is a point of Σ such that Σ is locally flat except possibly at p . Two sets or pairs of sets embedded in a manifold are *equivalent* (denoted by $\Leftrightarrow$) if there is a global homeomorphism carrying one set or pair of sets onto the other. Let $\mathcal{J}_*$ and $\mathcal{S}_*$ be the sets of equivalence classes of $\mathcal{J}$ and $\mathcal{S}$ in S^3 , respectively.

Let $(\alpha, p) \in \mathcal{J}$. Let Σ be any 2-sphere in S^3 such that Σ intersects α only at the endpoint which is not p and such that $\Sigma \cup \alpha$ is locally flat at every point of Σ . Let ϕ be a map of S^3 onto itself such that $\phi(\alpha) = p$ and $\phi|_{(S^3 - \alpha)}$ is a homeomorphism onto $S^3 - p$. Such a map exists by Theorem 1. Define $\Psi: \mathcal{J} \rightarrow \mathcal{S}$ by $\Psi(\alpha, p) = (\phi(\Sigma), p)$.

Let $(\Sigma, p) \in \mathcal{S}$. We noticed in §2 that one of the closed complementary domains G of Σ is a 3-cell. Let g be a homeomorphism of B^3 onto G such that $g(1) = p$. Define $\Gamma: \mathcal{S} \rightarrow \mathcal{S}$ by $\Gamma(\Sigma, p) = (g(I), p)$.

THEOREM 3. Ψ and Γ are well defined up to equivalence class and Ψ induces a one-to-one correspondence $\Psi_*: \mathcal{S}_* \rightarrow \mathcal{S}_*$ such that its inverse is Γ^* , the function induced by Γ .

We will generalize this result to 2-manifolds in a 3-manifold in Theorem 4. The proof of Theorem 3 will then follow from Theorem 4.

Now let N be some fixed 3-manifold. Let $\mathcal{A}$ be the collection of sets in N each of which is the union of a locally flat 2-manifold K and a set of disjoint arcs α_i , $i = 1, \dots, m$, such that α_i intersects K at one endpoint and $K \cup \alpha_i$ is locally flat except at the other endpoint for each i . Let $\mathcal{M}$ be the collection of nearly flat 2-manifolds M in N (i.e., M is wild at a finite number of points). Let $\mathcal{A}_*$ and $\mathcal{M}_*$ be the sets of equivalence classes of $\mathcal{A}$ and $\mathcal{M}$ in N , respectively.

Let $K \cup (\bigcup_1^m \alpha_i) \in \mathcal{A}$ and let p_i be the wild endpoint of α_i , $i = 1, \dots, m$. Let ϕ be a map of N onto itself such that $\phi(\alpha_i) = p_i$, $i = 1, \dots, m$, and $\phi|_{(N - \bigcup_1^m \alpha_i)}$ is a homeomorphism onto $N - \bigcup_1^m p_i$. The existence of such a map follows from Theorem 1. Define $\Psi: \mathcal{A} \rightarrow \mathcal{M}$ by $\Psi(K \cup (\bigcup_1^m \alpha_i)) = \phi(K)$.

Let $M \in \mathcal{M}$ and let p_i be the wild points of M , $i = 1, \dots, m$. Let $g_i: B^3 \rightarrow N$ be homeomorphisms with disjoint images such that:

- (1) $g_i(S_+^2) \subset M$,
- (2) $g_i(a) = p_i$,
- (3) $g_i(B^3 - S_+^2) \subset N - M$,
- (4) $g_i(S^2)$ is locally flat except at p ,
- (5) $K = (M - \bigcup_1^m g_i(P_0)) \cup (\bigcup_1^m g_i(D_0))$ is locally flat.

Let $\alpha_i = g_i(J)$, $i = 1, \dots, m$. Define $\Gamma: \mathcal{M} \rightarrow \mathcal{A}$ by $\Gamma(M) = K \cup (\bigcup_1^m \alpha_i)$.

THEOREM 4. Ψ and Γ are well defined up to equivalence class and Ψ induces a one-to-one correspondence $\Psi_*: \mathcal{A}_* \rightarrow \mathcal{M}_*$ such that its inverse is Γ_* , the function induced by Γ .

Proof. (i) Ψ is well defined up to equivalence class and induces a function $\Psi_*: \mathcal{A}_* \rightarrow \mathcal{M}_*$. Indeed, given the diagram with the solid arrows:

$$\begin{array}{ccc} K_1 \cup (\bigcup_1^m \alpha_i^1) & \xlongequal{\quad} & K_2 \cup (\bigcup_1^m \alpha_i^2) \\ \Psi \downarrow & & \downarrow \Psi \\ M_1 & \xlongequal{\quad} & M_2 \end{array}$$

where $K_1 \cup (\bigcup_1^m \alpha_i^1)$, $K_2 \cup (\bigcup_1^m \alpha_i^2) \in \mathcal{A}$ and $M_1, M_2 \in \mathcal{M}$, we will show that we can fill in the dotted arrow.

Let p_i^j be the wild endpoint of $\alpha_i^1, i = 1, \dots, m, j = 1, 2$. There is a homeomorphism

$$f: (N, K_1 \cup (\bigcup_1^m \alpha_i^1), \bigcup_1^m p_i^1) \rightarrow (N, K_2 \cup (\bigcup_1^m \alpha_i^2), \bigcup_1^m p_i^2).$$

Without loss of generality $f(\alpha_i^1) = \alpha_i, i = 1, \dots, m$. By the definition of Ψ , for each $j = 1, 2$, there is a map ϕ_j of N onto itself such that $\phi_j(\alpha_i^j) = p_i^j, i = 1, \dots, m, \phi_j|_{(N - \bigcup_1^m \alpha_i^j)}$ is a homeomorphism onto $N - \bigcup_1^m p_i^j$ and $\phi_j(K_j) = M_j$.

Define $g: N \rightarrow N$ by

$$g(x) = \begin{cases} \phi_2 f \phi_1^{-1}(x), & x \in N - \bigcup_1^m p_i^1, \\ p_i^2, & x = p_i^1, i = 1, \dots, m. \end{cases}$$

Evidently g is a homeomorphism and

$$\begin{aligned} g(M_1 - \bigcup_1^m p_i^1) &= \phi_2 f \phi_1^{-1}(M_1 - \bigcup_1^m p_i^1) \\ &= \phi_2 f(K_1 - \bigcup_1^m \alpha_i^1) \\ &= \phi_2(K_2 - \bigcup_1^m \alpha_i^2) \\ &= M_2 - \bigcup_1^m p_i^2. \end{aligned}$$

Thus $g(M_1) = M_2$ and so M_1 is equivalent to M_2 .

Now define $\Psi_*: \mathcal{A}_* \rightarrow \mathcal{M}_*$ by

$$\Psi_*[K \cup (\bigcup_1^m \alpha_i)] = [\Psi(K \cup (\bigcup_1^m \alpha_i))].$$

(ii) Γ is well defined up to equivalence class and induces a function

$$\Gamma_*: \mathcal{M}_* \rightarrow \mathcal{A}_*.$$

Indeed, given the diagram with the solid arrows:

$$\begin{array}{ccc} M_1 & \xleftarrow{\quad} & M_2 \\ \Gamma \downarrow & & \downarrow \Gamma \\ K_1 \cup (\bigcup_1^m \alpha_i^1) & \xleftarrow{\quad} & K_2 \cup (\bigcup_1^m \alpha_i^2) \end{array}$$

where $M_1, M_2 \in \mathcal{M}$ and $K_1 \cup (\bigcup_1^m \alpha_i^1), K_2 \cup (\bigcup_1^m \alpha_i^2) \in \mathcal{A}$, we will show that we can fill in the dotted arrow.

Let $p_i^j, i = 1, \dots, m$, be the wild points of $M_j, j = 1, 2$. There is a homeomorphism

$$h: (N, M_1, \bigcup_1^m p_i^1) \rightarrow (N, M_2, \bigcup_1^m p_i^2).$$

Without loss of generality $h(p_i^1) = p_i^2, i = 1, \dots, m$. By the definition of Γ , for each $j = 1, 2$, there are homeomorphisms $g_i^j: B^3 \rightarrow N$ with disjoint images such that

- (1) $g^j(S_+^2) \subset M_j$,
- (2) $g_i^j(a) = p_i^j$,
- (3) $g_i(B^3 - S_+^2) \subset N - M_j$,

- (4) $g_i^j(S^2)$ is locally flat except at p_i^j ,
 (5) $K_j = (M_j - \bigcup_1^m g_i^j(P_0)) \cup (\bigcup_1^m g_i^j(D_0))$ is locally flat,
 (6) $\alpha_i^j = g_i^j(J)$.

Let $q_i^j = g_i^j(0, 0, \frac{1}{2})$, $i = 1, \dots, m$, $j = 1, 2$.

Without loss of generality $h(g_i^1(D_0)) \subset g_i^2(G_0 - D_0)$. Let $f_i: B^3 \rightarrow g_i^2(B^3)$ be a homeomorphism such that:

- (7) $f_i(a) = p_i^2$,
 (8) $f_i(D_0) = h(g_i^1(D_0))$,
 (9) $f_i(B^2) = g_i^2(D_0)$,
 (10) $f_i(0) = q_i^2$.

Since $g_i^2(S^2)$ is locally flat except at p_i^2 , we can extend f_i to a homeomorphism $f_i: B_2(b) \rightarrow N$ such that the images are disjoint and

$$f_i^{-1}(\text{Cl}(M_2 - g_i^2(S_+^2))) \subset ((\text{Int } R_-^3) \cap B_2(b)).$$

Choose $\varepsilon > 0$ such that

$$f_i^{-1}(\text{Cl}(M_2 - g_i^2(S_+^2))) \subset \{(x, y, z) \in B_2(b) \mid z < -\varepsilon\}.$$

It follows from Lemma 3 that there is a homeomorphism r_i of N onto itself such that:

- (11) $r_i|(N - f_i(B_2(b))) = 1$,
 (12) $r_i(M_2) = M_2$,
 (13) $r_i(h(K_1) \cap f_i(B_2(b))) = K_2 \cap f_i(B_2(b))$,
 (14) $r_i(h(q_i^1)) = q_i^2$.

Now $r_i(h(\alpha_i^1))$ may not be equal to α_i^2 . However, $r_i h g_i^1$ and g_i^2 are homeomorphisms of G_0 onto $g_i^2(G_0)$ such that $r_i h g_i^1(a) = g_i^2(a) = p_i^2$ and $r_i h g_i^1(0, 0, \frac{1}{2}) = g_i^2(0, 0, \frac{1}{2}) = q_i^2$. Thus it follows from Lemma 5 that there is a homeomorphism s_i of N onto itself such that:

- (15) $s_i|(N - g_i^2(G_0)) = 1$,
 (16) $s_i r_i h g_i^1|J = g_i^2|J$.

Define $h_1: N \rightarrow N$ by $h_1 = s_m r_m \cdots s_1 r_1 h$. Then $h_1(K_1 \cup (\bigcup_1^m \alpha_i^1)) = K_2 \cup (\bigcup_1^m \alpha_i^2)$ so that $K_1 \cup (\bigcup_1^m \alpha_i^1)$ is equivalent to $K_2 \cup (\bigcup_1^m \alpha_i^2)$.

Now define $\Gamma_*: \mathcal{M}_* \rightarrow \mathcal{A}_*$ by $\Gamma_*[M] = [\Gamma(M)]$.

(iii) $\Gamma_* \Psi_* = 1$. Indeed, given the diagram with the solid arrows:

$$\begin{array}{ccc} K_1 \cup (\bigcup_1^m \alpha_i^1) & \xrightarrow{\Psi} & M \\ & \searrow \text{dotted} & \downarrow \Gamma \\ & & K_2 \cup (\bigcup_1^m \alpha_i^2) \end{array}$$

where $K_1 \cup (\bigcup_1^m \alpha_i^1), K_2 \cup (\bigcup_1^m \alpha_i^2) \in \mathcal{A}$ and $M \in \mathcal{M}$, we will show that we can fill in the dotted arrow.

Let p_i be the wild endpoint of α_i^1 and let q_i be the other endpoint, $i = 1, \dots, m$. Since $K_1 \cup \alpha_i^1$ is locally flat at q_i , there are disjoint neighborhoods W_i of the q_i in N and homeomorphisms

$$h_i: (R^3, (R^2 \times 0) \cup (0 \times R_+^1)) \rightarrow (W_i, W_i \cap (K_1 \cup \alpha_i^1)),$$

$i = 1, \dots, m$. Let $\beta_i = h_i(0 \times B_-^1)$. There is a homeomorphism f_i of N onto itself such that:

- (1) $f_i(\alpha_i^1 \cup \beta_i) = \alpha_i^1$,
- (2) $f_i h_i(S_-^2) = h_i(B_-^2)$,
- (3) $f_i h_i(B_-^2) = h_i(S_+^2)$,
- (4) $f_i|((N - W_i) \cup (K_1 - h_i(B_-^2))) = 1$.

Choose $\varepsilon > 0$ sufficiently small that ε -neighborhoods of the $f_i(\alpha_i^1)$'s are disjoint and do not intersect K_1 . By Theorem 1 there is a map ϕ_i of N onto itself such that:

- (5) $\phi_i f_i(\alpha_i^1) = p_i$,
- (6) $\phi_i f_i(\beta_i) = \alpha_i^1$,
- (7) $\phi_i| (N - V_\varepsilon f_i(\alpha_i^1)) = 1$,
- (8) $\phi_i| (N - f_i(\alpha_i^1))$ is a homeomorphism onto $N - p_i$.

Then $\phi = \phi_m f_m \cdots \phi_1 f_1$ is a map of N onto itself such that:

- (9) $\phi(\alpha_i^1) = p_i$, $i = 1, \dots, m$,
- (10) $\phi| (N - \bigcup_1^m \alpha_i^1)$ is a homeomorphism onto $N - \bigcup_1^m p_i$.

Let $M_1 = \phi(K_1)$. Then $M_1 \in \Psi_*[K_1 \cup (\bigcup_1^m \alpha_i^1)]$ and so by (i) M_1 is equivalent to M .

Define $e: B^3 \rightarrow B_-^3$ by a canonical push down so that $e(0 \times B^1) = 0 \times B_-^1$. Define $g_i: B^3 \rightarrow N$ by $g_i = \phi_i f_i h_i e$. Then it is easy to see that the images of the g_i are disjoint and that g_i satisfies the properties:

- (11) $g_i(S_+^2) \subset M_1$,
- (12) $g_i(B^3 - S_+^2) \subset N - M_1$,
- (13) $g_i(S_-^2)$ is locally flat except at p_i ,
- (14) $K_1 = (M_1 - \bigcup_1^m g_i(S_+^2)) \cup (\bigcup_1^m g_i(S_-^2))$ is locally flat,
- (15) $\alpha_i^1 = g_i(0 \times B^1)$.

Thus $K_1 \cup (\bigcup_1^m \alpha_i^1) \in \Gamma[M_1]$ and so by (ii) $K_1 \cup (\bigcup_1^m \alpha_i^1)$ is equivalent to $K_2 \cup (\bigcup_1^m \alpha_i^2)$. Hence $\Gamma_* \Psi_* = 1$.

(iv) $\Psi_* \Gamma_* = 1$. Indeed, given the diagram with the solid arrows:

$$\begin{array}{ccc} M_1 & \xrightarrow{\Gamma} & K \cup (\bigcup_1^m \alpha_i) \\ & \swarrow \text{dotted arrow} \searrow & \downarrow \Psi \\ & & M_2 \end{array}$$

where $M_1, M_2 \in \mathcal{M}$ and $K \cup (\bigcup_1^m \alpha_i) \in \mathcal{A}$, we will show that we can fill in the dotted arrow.

Let p_i be the wild points of M_1 , $i = 1, \dots, m$. By the definition of Γ there are homeomorphisms $g_i: B^3 \rightarrow N^3$ with disjoint images such that:

- (1) $g_i(S_+^2) \subset M_1$,
- (2) $g_i(a) = p_i$,
- (3) $g_i(B^3 - S_+^2) \subset N - M_1$,
- (4) $g_i(S^2)$ is locally flat except at p_i ,
- (5) $K = (M_1 - \bigcup_1^m g_i(P_0)) \cup (\bigcup_1^m g_i(D_0))$ is locally flat,
- (6) $\alpha_i = g_i(J)$.

Let $h_i: B^3 \rightarrow g_i(B^3)$ be a homeomorphism such that:

- (7) $h_i(0 \times B_+^1) = \alpha_i$,
- (8) $h_i(S_+^2) = g_i(P_0)$,
- (9) $h_i(B^2) = g_i(D_0)$.

Since $g_i(S^2)$ is locally flat except at p_i , we can extend h_i to a homeomorphism $h_i: B_2(b) \rightarrow N$ such that the images are disjoint and

$$h_i^{-1}(M_1 - g_i(P_0)) \subset R_-^3 \cap (B_2(b) - V_1(0)).$$

It follows from Lemma 4 that there is a map ϕ_i of N onto itself such that:

- (10) $\phi_i|(N - h_i(B_2(b))) = 1$,
- (11) $\phi_i(\alpha_i) = p_i$,
- (12) $\phi_i(K \cap h_i(B_2(b))) = M_1 \cap h_i(B_2(b))$,
- (13) $\phi_i|(N - \alpha_i)$ is a homeomorphism onto $N - p_i$.

Then $\phi = \phi_m \cdots \phi_1$ is a map of N onto itself such that:

- (14) $\phi(\alpha_i) = p_i$, $i = 1, \dots, m$,
- (15) $\phi|(N - \bigcup_1^m \alpha_i)$ is a homeomorphism onto $N - \bigcup_1^m p_i$,
- (16) $\phi(K) = M_1$.

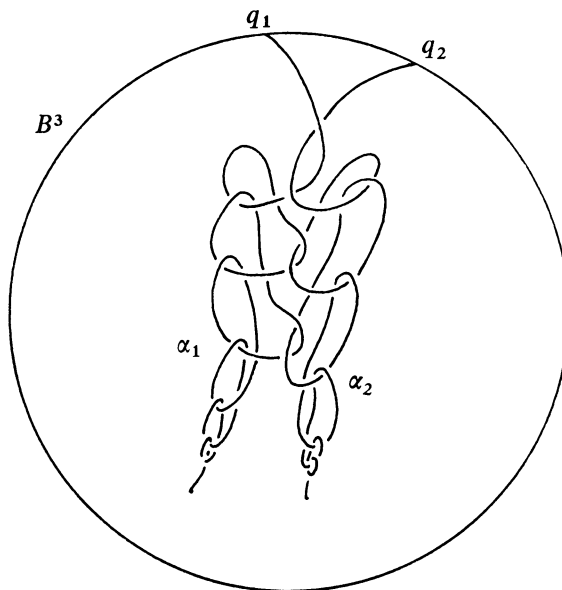
Thus $M_1 \in \Psi_*[K \cup (\bigcup_1^m \alpha_i)]$ and so by (i) M_1 is equivalent to M_2 . Hence $\Psi_*\Gamma_* = 1$.

Proof of Theorem 3. Let $\mathcal{A}$ be the set of pairs (Σ, α) where Σ is a flat 2-sphere in S^3 , α is an arc which intersects Σ at one end point and $\Sigma \cup \alpha$ is locally flat except at the other endpoint p and let $\mathcal{A}_*$ be the sets of equivalence classes of $\mathcal{A}$ in S^3 . By Lemma 7 the mapping $(\Sigma, \alpha) \rightarrow (\alpha, p)$ induces a one-to-one correspondence between $\mathcal{A}_*$ and $\mathcal{S}_*$. By Theorem 4 there is a one-to-one correspondence between $\mathcal{S}_*$ and $\mathcal{A}_*$ and the composition of these two is the desired one-to-one correspondence between $\mathcal{S}_*$ and $\mathcal{S}_*$.

4. Uniqueness of a decomposition space.

THEOREM 5. Let $\alpha_i^1, i = 1, \dots, m$, be disjoint arcs in B^3 and let $\alpha_i^2, i = 1, \dots, m$, be disjoint arcs in B^3 such that α_i^1 intersects S^2 at one endpoint and $S^2 \cup \alpha_i^2$ is locally flat in R^3 except at the other endpoint $p_i^2, i = 1, \dots, m, j = 1, 2$. For each $j = 1, 2$, let H_j be the decomposition space of B^3 whose nondegenerate elements are the arcs $\alpha_i^j, i = 1, \dots, m$. If H_1 is homeomorphic to H_2 , then, with a suitable ordering of the α_i^j 's, there is a homeomorphism of B^3 onto itself carrying α_i^1 onto $\alpha_i^2, i = 1, \dots, m$.

Before proving the theorem, let us consider the example illustrated in the following figure:



It is impossible to find disjoint closed 3-cells G_i in B_3 such that $\alpha_i - q_i \subset \text{Int } G_i$, $i = 1, 2$, so that it looks like the theorem may require a global proof. However, these arcs are cellular in R^3 by Theorem 1 so that there are disjoint Euclidean neighborhoods of the arcs in R^3 . Thus we see that there may be a local proof for the theorem. In fact, the proof follows from Theorem 4 which has a local proof.

Proof. Without loss of generality, for each $j = 1, 2$, H_j may be considered as a subset of R^3 which is the image of B^3 under a decomposition map of R^3 , i.e., there is a map ϕ_j of R^3 onto itself such that:

- (1) $\phi_j(\alpha_i^j) = p_i^j$, $i = 1, \dots, m$,
- (2) $\phi_j|_{(R^3 - \bigcup_1^m \alpha_i^j)}$ is a homeomorphism onto $R^3 - \bigcup_1^m p_i^j$,
- (3) $\phi_j(B^3) = H_j$.

Since $\text{Cl}(R^3 - H_j) \approx \text{Cl}(R^3 - B^3)$, $j = 1, 2$, we can extend the homeomorphism of H_1 onto H_2 to a homeomorphism of R^3 onto itself. Thus $\phi_1(S^2)$ is equivalent to $\phi_2(S^2)$ in R^3 . Since $\phi_j(S^2) = \Psi(S^2 \cup (\bigcup_1^m \alpha_i^j))$, $j = 1, 2$, it follows from Theorem 4 that $S^2 \cup (\bigcup_1^m \alpha_i^1)$ is equivalent to $S^2 \cup (\bigcup_1^m \alpha_i^2)$. Thus there is a homeomorphism $h: R^3 \rightarrow R^3$ such that $h(S^2 \cup (\bigcup_1^m \alpha_i^1)) = S^2 \cup (\bigcup_1^m \alpha_i^2)$. Without loss of generality $h(\alpha_i^1) = \alpha_i^2$, $i = 1, \dots, m$. Then $h|_{B^3}$ is the required homeomorphism.

5. Characterization of a class of crumpled cubes. A *crumpled n -cube* is a topological space which is homeomorphic to a closed complementary domain of an $(n - 1)$ -sphere embedded in the n -sphere S^n .

THEOREM 6. Let H be a crumpled n -cube in S^n such that $G = \text{Cl}(S^n - H)$ is a

closed n -cell. Then H is homeomorphic to a decomposition space of B^n whose nondegenerate elements are arcs which intersect S^{n-1} at one endpoint and are locally flat except possibly at the other endpoint. Moreover, these arcs correspond to the singular points of $\Sigma = \text{Bd}H$ (i.e., the points at which Σ is not locally flat).

Notice that for $n \geq 4$ it follows from [19], [10] that the arcs are locally flat at every point. However, for $n = 3$ the arcs may or may not be locally flat at the endpoint. Also for $n = 3$ it follows from [20], [22] that any crumpled 3-cube can be embedded in S^3 so that the closure of the complement is a closed 3-cell. Thus the conclusion of Theorem 6 holds for any crumpled 3-cube.

Proof. Let g be a homeomorphism of B^n onto G . Let a point $p \in B^n$ be represented by the coordinates (u, x) where u is the distance from p to the origin and x is the point of S^{n-1} which lies on the ray from the origin through p . Let X be the set of singular points of Σ , let $X' = g^{-1}(X)$ and let μ be a map from S^{n-1} into I such that $\mu(X') = 1$ and $\mu(S^{n-1} - X') \subset I'$. Define a map $\theta: B^n \rightarrow B^n$ as follows:

$$\theta(1/2, x) = (1/2 + 1/2\mu(x), x),$$

$$\theta \text{ maps } [0, (1/2, x)] \text{ linearly onto } [0, \theta(1/2, x)],$$

$$\theta \text{ maps } [(1/2, x), (1, x)] \text{ linearly onto } [\theta(1/2, x), (1, x)].$$

Define a map $\phi: S^n \rightarrow S^n$ by

$$\phi(p) = \begin{cases} g\theta g^{-1}(p), & p \in G, \\ p, & p \in H. \end{cases}$$

Now $H - X$ is a manifold with boundary $\Sigma - X$ and $\phi g([1/2, 1] \times (S^{n-1} - X'))$ is a closed collar attached to $H - X$. Thus by Theorem 2 there is a homeomorphism

$$h_1: (H - X) \cup \phi g([1/2, 1] \times (S^{n-1} - X')) \rightarrow H - X.$$

Let H_1 be the closed complementary domain of $g(\text{Bd}B_{1/2}(0))$ which contains Σ . Then we can extend h_1 to a homeomorphism $h_1: \phi(H_1) \rightarrow H$ via the identity on X . Let h_2 be a homeomorphism from B^n onto H_1 . Then $h = h_1\phi h_2$ is the required map of B^n onto H . For, if $x \in X$, then $h^{-1}(x) = h_2^{-1}g([1/2, 1] \times g^{-1}(x))$, an arc which intersects S^{n-1} at one endpoint and is locally flat except possibly at the other endpoint, and if $x \in H - X$, then $h^{-1}(x)$ is a single point.

6. Characterization of pseudo-half spaces. In this section we will characterize pseudo-half spaces. First we state a lemma.

LEMMA 8. *If $(X, Y) \approx (R_+^n, R^{n-1})$ and $X \cup p$ is the one-point compactification of X , then $(X \cup p, Y \cup p) \approx (B^n, S^{n-1})$.*

THEOREM 7. *M is an n -pseudo-half space if and only if $M \approx B^n - \alpha$ where α is an arc in B^n such that α intersects S^{n-1} at one endpoint and $S^{n-1} \cup \alpha$ is locally flat except possibly at the other endpoint.*

Proof. Assume M is an n -pseudo-half space. By Theorem 2 we can add an open collar $\text{Bd}M \times [0, 1)$ to M by identifying $(x, 0)$ with x for $x \in \text{Bd}M$, so that $M \cup (\text{Bd}M \times [0, 1)) \approx \text{Int}M$. Without loss of generality the one-point compactification $M \cup (\text{Bd}M \times [0, 1)) \cup p$ is equal to S^n . By Lemma 8 there is a homeomorphism:

$$f: ((\text{Bd}M \times [0, 1)) \cup p, (\text{Bd}M \times 0) \cup p) \rightarrow (B^n, S^{n-1})$$

such that $f(p) = 1$. Let $B' = B_{1/4}(-1/4, 0, 0)$ and let $S' = \text{Bd}B'$. Now $f^{-1}(\text{Cl}(B^n - B') - I)$ is a closed collar attached to M . By Theorem 2 there is a homeomorphism

$$\begin{aligned} h: M &\rightarrow M \cup f^{-1}(\text{Cl}(B^n - B') - I) \\ &= \text{Cl}(S^n - f^{-1}(B')) - f^{-1}(I). \end{aligned}$$

Now $f^{-1}(S')$ is bi-collared in S^n and hence flat. Thus there is a homeomorphism $g: S^n \rightarrow S^n$ such that $g(\text{Cl}(S^n - f^{-1}(B'))) = B^n$. Let $\alpha = gf^{-1}(I)$. Then we have:

$$\begin{aligned} gh(M) &= g(\text{Cl}(S^n - f^{-1}(B')) - f^{-1}(I)) \\ &= g(\text{Cl}(S^n - f^{-1}(B'))) - gf^{-1}(I) \\ &= B^n - \alpha. \end{aligned}$$

It is evident that α has the required properties.

Assume $M \approx B^n - \alpha$ where α is an arc in B^n which intersects S^{n-1} at one endpoint q and is locally flat except at the other endpoint p . We can identify S^n with the one-point compactification of R^n . It is easy to show that $\text{Int}(B^n - \alpha) \approx S^n - \alpha$ by shrinking $\text{Cl}(S^n - B^n)$ to q . By Theorem 1 there is a map $g: S^n \rightarrow S^n$ such that $g(\alpha) = p$ and $g|(S^n - \alpha)$ is a homeomorphism onto $S^n - p$. Thus

$$\begin{aligned} \text{Int}M &\approx \text{Int}(B^n - \alpha) \approx S^n - \alpha \approx S^n - p \approx R^n, \\ \text{Bd}M &\approx \text{Bd}(B^n - \alpha) = S^{n-1} - q \approx R^{n-1}. \end{aligned}$$

Hence M is an n -pseudo-half space.

REMARK. We have actually proved that $B^n - \alpha$ is an n -pseudo-half space even if $S^{n-1} \cup \alpha$ is not locally flat at $S^{n-1} \cap \alpha$.

COROLLARY [CANTRELL, DOYLE]. For $n \neq 3$, $M \approx R^n_+$.

Proof. The proof is essentially that of Cantrell [7] as pointed out by Doyle [11] which we include for completeness. It follows from Theorem 2.1 of [16], a generalization of a theorem of Homma [19], that for $n > 3$ an arc in R^n which is locally

flat except at one endpoint is equivalent to an arc which is locally polyhedral except at one endpoint. By [10] the arc is locally flat at every point. For $n < 3$, this is true for every arc. So by Theorem 1 there is a map $g: B^n \rightarrow B^n$ such that $g|_{S^{n-1}} = 1$, $g(\alpha) = q$, and $g|(B^n - \alpha)$ is a homeomorphism onto $B^n - q$. Hence for $n \neq 3$, $M \approx B^n - \alpha \approx B^n - q \approx R_+^n$.

THEOREM 8. *If α_1 and α_2 are two arcs in B^3 which are not equivalent in R^3 such that α_1 intersects S^2 at one endpoint q_1 and $\alpha_i \cup S^2$ is locally flat in S^3 except possibly at the other endpoint p_i , $i = 1, 2$, then $B^3 - \alpha_1$ and $B^3 - \alpha_2$ are topologically different.*

Proof. Suppose we have $h: B^3 - \alpha_1 \approx B^3 - \alpha_2$. We can identify S^3 with the one-point compactification of R^3 and extend h to $h: S^3 - \alpha_1 \approx S^3 - \alpha_2$. By Theorem 1 there is a map $g_i: S^3 \rightarrow S^3$ such that $g_i(\alpha_i) = p_i$ and $g_i|(S^3 - \alpha_i)$ is a homeomorphism onto $S^3 - p_i$, $i = 1, 2$. Let $\Sigma_i = g_i(S^2)$, $i = 1, 2$, and define $f: S^3 \rightarrow S^3$ by:

$$f(x) = \begin{cases} g_2 h g_1^{-1}(x), & x \in S^3 - p_1, \\ p_2, & x = p_1. \end{cases}$$

Evidently f is a homeomorphism. Now

$$f(\Sigma_1 - p_1) = g_2 h g_1^{-1}(\Sigma_1 - p_1) = g_2 h(S^2 - q_1) = g_2(S^2 - q_2) = \Sigma_2 - p_2$$

so that $f(\Sigma_1) = \Sigma_2$. Thus Σ_1 is equivalent to Σ_2 . By Theorem 3, α_1 is equivalent to α_2 , a contradiction. Hence $B^3 - \alpha_1 \not\approx B^3 - \alpha_2$.

COROLLARY. *There are uncountably many topologically different 3-pseudo-half spaces.*

Proof. By [14] there are uncountably many inequivalent arcs in R^3 which are locally flat except at one endpoint.

THEOREM 9. *Let M_1 and M_2 be 3-pseudo-half spaces with common boundary F and disjoint interiors such that $M_1 \not\approx R_+^3$. Then $M_1 \cup M_2 \approx R^3$ if and only if $M_2 \approx R_+^3$.*

Proof. Assume $M_1 \cup M_2 = R^3$. We can identify S^3 with the one-point compactification $R^3 \cup p$ of R^3 . Then $F \cup p$ is a 2-sphere in S^3 which is locally flat except at p . Since $M_1 \cup p \not\approx B^3$, by [18] $M_2 \cup p \approx B^3$. Thus $M_2 \approx R_+^3$.

The converse follows immediately from Theorem 2.

COROLLARY. *If M is a 3-pseudo-half space such that $M \not\approx R_+^3$, then $M \times I \not\approx R_+^4$.*

Proof. $\text{Bd}(M \times I) = (\text{Bd } M \times I) \cup (M \times \text{Bd } I) \approx R^3$.

7. Cellularity of arcs in S^3 .

THEOREM 10. *If α is an arc in S^3 such that α contains a subarc β both of whose endpoints are isolated wild points of β , then α is not cellular.*

Proof. Suppose α is cellular.

Case 1. α is only wild at its endpoints a and b . Since α is cellular, there is a homeomorphism $h: S^3 - \alpha \rightarrow S^3 - p$ for some point $p \in S^3$. Let $q \in \text{Int } \alpha$. Since α is locally flat at q , there is an open 2-cell D in S^3 such that $D \cap \alpha = q$ and $D \cup \alpha$ is locally flat at every point of D . Then $D_1 = h(D - q) \cup p$ is an open 2-cell in S^3 which is locally flat except at p . It follows from Lemma 2 that there is an open 2-cell $D_2 \subset D_1$ such that $p \in D_2$ and D_2 is contained in a 2-sphere Σ_2 which is locally flat except at p . Then $\Sigma = h^{-1}(\Sigma_2 - p) \cup q$ is locally flat at every point and hence flat in S^3 and $\Sigma \cap \alpha = q$.

Let G_1 and G_2 be the closed complementary domains of Σ in S^3 and let $M_i = G_i - \alpha$, $i = 1, 2$. By Theorems 7 and 8, M_i is a 3-pseudo-half space but not R_+^3 , $i = 1, 2$. But $M_1 \cup M_2 = S^3 - \alpha \approx R^3$, which contradicts Theorem 9.

Case 2. α is wild at both endpoints a and b and at one interior point d . If $x, y \in \alpha$, let $\langle x, y \rangle$ denote the subarc of α from x to y . By [23], for $n \neq 4$, every subarc of a cellular arc is cellular. Thus $\langle a, d \rangle$ and $\langle d, b \rangle$ are both cellular and if either one is wild at both endpoints, we get a contradiction by Case 1. Hence suppose both $\langle a, d \rangle$ and $\langle d, b \rangle$ are locally flat at d . By [25] there is a neighborhood U of $\alpha - a$ such that every arc in $U \cup a$ with a as an endpoint is wild. By Theorem 1 there is a map $\phi: S^3 \rightarrow S^3$ such that $\phi\langle a, d \rangle = a$, $\phi|_{(S^3 - U)} = 1$ and $\phi|_{(S^3 - \langle a, d \rangle)}$ is a homeomorphism onto $S^3 - a$. Thus $\phi\langle d, b \rangle$ is cellular and wild at both endpoints. Again we get a contradiction by Case 1.

General Case. Let γ be a subarc of β such that γ contains all the wild points of β except its endpoints. Then β and γ are both cellular. Thus there is a map $\phi: S^3 \rightarrow S^3$ such that $\phi(\gamma)$ is a point and $\phi|_{(S^3 - \gamma)}$ is a homeomorphism. Then $\phi(\beta)$ reduces to either Case 1 or Case 2 and we get a contradiction. Hence α is not cellular.

The following theorem is a special case of Theorem 1 of [12]. However, the proof here does not use the axiom of choice.

THEOREM 11 (DOYLE). *If α is an arc in S^3 such that α contains no subarc both of whose endpoints are wild, then α is cellular.*

Proof. Let p and r be the endpoints of α . There is a natural ordering, denoted by $<$, of the points of α from p to r . If β and γ are subarcs of α , we will say that $\beta < \gamma$ if $x < y$ for arbitrary $x \in \text{Int } \beta$ and $y \in \text{Int } \gamma$.

Let X be the set of wild points of α . Then X is countable since it has the same order as the set of components of $\alpha - X$. There is at most one point q of X such that q does not lie on some flat subarc of α and with no loss of generality such a q exists.

Let $\varepsilon > 0$. Since X is a countable compact set, there is a flat closed 3-cell $E \subset V_\varepsilon(\alpha)$ such that $X \subset \text{Int } E$.

Let $\mathcal{U}_0$ be the collection of closed complementary domains of $X \cap \langle p, q \rangle$ in $\langle p, q \rangle$ which are not contained in $\text{Int } E$ and let $\beta_1 = \langle y_1, z_1 \rangle$ be the last such arc in $\mathcal{U}_0$. Let $y'_1 \in \text{Int } \beta_1$ such that $\langle y'_1, z_1 \rangle \subset \text{Int } E$. Let h_1 be a homeomorphism of S^3 onto itself such that:

- (1) $h_1|_{((S^3 - V_\varepsilon(\alpha)) \cup \langle z_1, r \rangle)} = 1$,
- (2) $h_1(\beta_1) = \langle y'_1, z_1 \rangle$,
- (3) $h_1(X) \subset \text{Int } E$.

Then $F_1 = h_1^{-1}(E)$ is a flat closed 3-cell in $V_\varepsilon(\alpha)$ such that $X \cup \langle y_1, q \rangle \subset \text{Int } F_1$.

Let $\mathcal{U}_1$ be the collection of closed complementary domains of $X \cap \langle p, q \rangle$ in $\langle p, q \rangle$ which are not contained in $\text{Int } F_1$ and let $\beta_2 = \langle y_2, z_2 \rangle$ be the last such arc in $\mathcal{U}_1$. As before we construct a flat closed 3-cell F_2 in $V_\varepsilon(\alpha)$ such that $X \cup \langle y_2, q \rangle \subset \text{Int } F_2$.

If this process continued indefinitely, we would get a sequence of points $y_i \in X$ with $y_{i+1} < y_i$ and a sequence of flat closed 3-cells F_i such that

$$X \cup \langle y_i, q \rangle \subset \text{Int } F_i, \quad i = 1, 2, \dots$$

Then $y = \lim_{i \rightarrow \infty} y_i$ would be an element of X such that $y \neq q$ and y is contained in no flat subarc of α , a contradiction. Hence the process must end, i.e., there is a flat closed 3-cell $F \subset V_\varepsilon(\alpha)$ such that $X \cup \langle p, q \rangle \subset \text{Int } F$.

Similarly, we can start at the other end of α and construct a flat closed 3-cell $F' \subset V_\varepsilon(\alpha)$ such that $\alpha \subset \text{Int } F'$. Since ε is arbitrary, α is cellular.

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